A soluble superconductive glass model

1993 J. Phys. A: Math. Gen. 26 L1201

(http://iopscience.iop.org/0305-4470/26/23/002)

J. Phys. A: Math. Gen. 26 (1993) L1201–L1205. Printed in the UK

LETTER TO THE EDITOR

A soluble superconductive glass model

D Sherrington and M Simkin†

Department of Physics, Theoretical Physics, 1 Keble Road, Oxford OX1 3NP, England

Received 26 August 1993

Abstract. An infinite-range superconductive glass model is introduced and solved exactly within replica-symmetric theory.

There has been an interest in recent years in the glassy behaviour of granular superconductors in magnetic fields [1–3]. In theoretical papers on this subject numerical or approximate methods were exploited.

The aim of the present letter is to introduce an infinite-range model, related to this physical system, which can be solved exactly.

Let us first consider an array of N superconducting grains coupled by Josephson tunnelling. The ith cluster is described by the phase ϕ_i of the wavefunction of Cooper pairs in it. This array in a magnetic field is described by the Hamiltonian

$$H=-\sum_{i<j}\tilde{J}_{ij}\cos(\phi_i-\phi_j-A_{ij}) \tag{1}$$

where

$$A_{ij}=\frac{2\pi}{\Phi_0}\int_i^j A\,\mathrm{d}l. \tag{2}$$

Here A is the vector potential of the magnetic field, and Φ_0 is the magnetic flux quantum. Applying the magnetic field $\boldsymbol{H}$ in the z direction, we have:

$$A_{ij}=\frac{2\pi}{\Phi_0}\,\frac{x_i+x_j}{2}(y_i-y_j) \tag{3}$$

where x_i, y_i are the Cartesian space coordinates of the ith grain. Thus we see that site disorder can lead to frustration in this model.

Let us now turn to our model, stimulated by the above, the conventional gauge glass [4] and the infinite-ranged spin-glass [5]. We consider the case when the coupling energies between any pair of superconducting grains are equal: $\tilde{J}_{ij}=\tilde{J}$. Next we assume that the A_{ij} are independently distributed with the Gaussian probability distribution

$$P(A_{ij})=(2\pi\sigma)^{-1/2}\exp(-A_{ij}^2/2\sigma). \tag{4}$$

† Present address: Frank Laboratory of Neutron Physics, Joint Institute for Nuclear Research, Moscow Region, 141980 Dubna, Russia.

0305-4470/93/231201+05 $07.50

In order to have both superconductive and superconductive-glass phases we require the following scaling of the coupling energy $\tilde{J}$ and dispersion σ with the number of grains N:

$$\tilde{J}=J/N^{1/2}$$
$$\sigma=\ln(N/\kappa^2). \tag{5}$$

To calculate the average of the free energy over the disorder we use the replica trick [6], which, by exploiting the mathematical identity

$$\ln Z=\lim_{n\to 0}\frac{Z^n-1}{n}=\lim_{n\to 0}\frac{1}{n}\ln Z^n \tag{6}$$

permits us to average Z^n instead of $\ln Z$:

$$F=[\ln Z]=\lim_{n\to 0}\frac{[Z^n]-1}{n}=\lim_{n\to 0}\frac{1}{n}\ln [Z^n]. \tag{7}$$

Here the [ ] brackets denote an average over the A_{ij} distribution. Introducing a replica label $\alpha=1,\ldots,n$, Z^n may be rewritten as

$$Z^n=\mathrm{Tr}\exp\left\{\frac{J}{N^{1/2}T}\sum_{i<j;\,\alpha}\cos(\phi_i^\alpha-\phi_j^\alpha-A_{ij})\right\}. \tag{8}$$

After performing the disorder average using the distribution of A_{ij}, given by (4), and keeping only intensive terms, we obtain

$$[Z^n]=\mathrm{Tr}\exp\left\{\frac{1}{N}\left(\sum_{i>j}\left[\frac{\kappa J}{T}\sum_\alpha\cos(\phi_i^\alpha-\phi_j^\alpha)+\frac{J^2}{4T^2}\sum_{\alpha\beta}\cos(\phi_i^\alpha-\phi_i^\beta-\phi_j^\alpha+\phi_j^\beta)\right]\right)\right\}. \tag{9}$$

Introducing vector variables, as in [7],

$$S_i^\alpha=(\cos\phi_i^\alpha,\sin\phi_i^\alpha)$$
$$S_i^{\alpha\beta}=(\cos(\phi_i^\alpha-\phi_i^\beta),\sin(\phi_i^\alpha-\phi_i^\beta)) \tag{10}$$

we rewrite (9) as

$$[Z^n]=\mathrm{Tr}\exp\left\{\frac{1}{2N}\sum_{ij}\left[\frac{\kappa J}{T}\sum_\alpha S_i^\alpha\cdot S_j^\alpha+(J/2T)^2\sum_{\alpha\beta}S_i^{\alpha\beta}\cdot S_j^{\alpha\beta}\right]\right\}. \tag{11}$$

This can be reduced to a form in which the exponents are linear in S via the identity

$$\exp(\lambda|\boldsymbol{a}|^2)=\frac{1}{2\pi}\int \mathrm{d}\boldsymbol{r}\exp\left(-\frac{|\boldsymbol{r}|^2}{2}+(2\lambda)^{1/2}\boldsymbol{a}\cdot\boldsymbol{r}\right). \tag{12}$$

Thus we transform (11) to

$$[Z^n]=\exp(\tfrac{1}{2}nN(J/2T)^2)\int\prod_\alpha\frac{N^{1/2}\,\mathrm{d}\boldsymbol{x}^\alpha}{2\pi}\prod_{(\alpha\beta)}\frac{N^{1/2}\,\mathrm{d}\boldsymbol{y}^{(\alpha\beta)}}{2\pi}$$
$$\times\mathrm{Tr}\exp\left\{-N\sum_\alpha\frac{(\boldsymbol{x}^\alpha)^2}{2}-N\sum_{(\alpha\beta)}\frac{(\boldsymbol{y}^{(\alpha\beta)})^2}{2}+(\kappa J/T)^{1/2}\sum_\alpha\left(\sum_i S_i^\alpha\right)\cdot\boldsymbol{x}^\alpha\right.$$
$$\left.+\frac{J}{2T}\sum_{(\alpha\beta)}\left(\sum_i S_i^{\alpha\beta}\right)\cdot\boldsymbol{y}^{(\alpha\beta)}\right\}. \tag{13}$$

Equation (13) can now be reduced to involve only a single site trace:

$$[Z^n]=\exp(\tfrac{1}{2}nN(J/2T)^2)\int\prod_{\alpha}\frac{N^{1/2}\,\mathrm{d}x^\alpha}{2\pi}\prod_{(\alpha\beta)}\frac{N^{1/2}\,\mathrm{d}y^{(\alpha\beta)}}{2\pi}$$
$$\times\exp\left\{-N\sum_{\alpha}\frac{|(x^\alpha)|^2}{2}+\sum_{(\alpha\beta)}\frac{|(y^{(\alpha\beta)})|^2}{2}-\ln\operatorname{Tr}\exp\left[(\kappa J/T)^{1/2}\sum_{\alpha}S^\alpha\cdot x^\alpha\right.\right.$$
$$\left.\left.+\frac{J}{2T}\sum_{(\alpha\beta)}S_i^{\alpha\beta}\right)\cdot y^{(\alpha\beta)}\right]\Bigg)\Bigg\}. \tag{14}$$

For large N the integral is dominated by the region of maximum integrand and can be performed by the method of steepest descents.

In replica-symmetric approximation we assume that at the maximum all the x^α are equal as also are all $y^{(\alpha\beta)}$:

$$(x^\alpha)_{\max}=x_n \qquad (y^{(\alpha\beta)})_{\max}=y_n. \tag{15}$$

So at the maximum, the trace in (14) becomes

$$\operatorname{Tr}\exp\left[(\kappa J/T)^{1/2}x_n\sum_{\alpha}\cos\phi^\alpha+\frac{J}{2T}y_n\sum_{(\alpha\beta)}\cos(\phi^\alpha-\phi^\beta)\right]. \tag{16}$$

Rewriting

$$\sum_{(\alpha\beta)}\cos(\phi^\alpha-\phi^\beta)=\left|\left(\sum_{\alpha}S^\alpha\right)\right|^2-n \tag{17}$$

where

$$S^\alpha=(\cos\phi^\alpha,\sin\phi^\alpha) \tag{18}$$

and performing transformation as in (12), we rewrite (16) as

$$\exp\left(-n\frac{J}{2T}y_n\right)\times\operatorname{Tr}\frac{1}{2\pi}\int\mathrm{d}r\exp\left[(\kappa J/T)^{1/2}x_n\sum_{\alpha}\cos\phi^\alpha-\frac{r^2}{2}+(y_nJ/T)^{1/2}\sum_{\alpha}S^\alpha\cdot r\right]. \tag{19}$$

After performing the trace, equation (19) reduces to

$$\exp\left(-n\frac{J}{2T}y_n\right)\frac{1}{2\pi}\int r\,\mathrm{d}r\,\mathrm{d}\theta\exp(-r^2/2)\mathrm{I}_0^n(\Xi_n) \tag{20}$$

where

$$\Xi_n=\left(\frac{\kappa J}{T}x_n^2+\frac{J}{T}y_nr^2+2\kappa^{1/2}\frac{J}{T}x_ny_n^{1/2}r\cos\theta\right)^{1/2}. \tag{21}$$

Here I_0 is the zero order modified Bessel function.

Making the substitution

$$m_n=x_n(T/\kappa J)^{1/2}$$
$$q_n=y_n2T/J \tag{22}$$

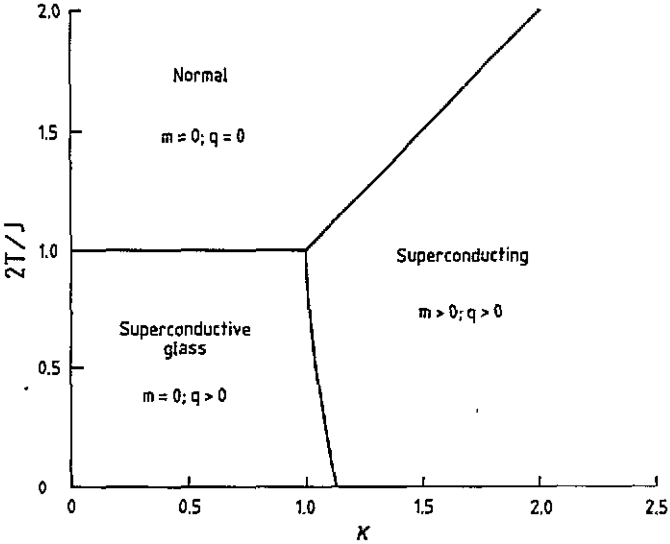

Figure 1. Phase diagram of the infinite-ranged superconductive glass model. In the chosen plane ($\kappa - T$) it is similar (but not identical) to that of the Sherrington–Kirkpatrick [5] spin-glass model.

applying the extremal condition to (13) and taking $\lim_{n\to 0}$, we obtain a set of equations for order parameters:

$$q = 1 - \left(\frac{2T}{J}\right)^2 \frac{\mathrm{d}}{\mathrm{d}q} \int r\,\mathrm{d}r \frac{\mathrm{d}\theta}{2\pi} \exp(-r^2/2) \ln \mathrm{I}_0(\Xi)$$

$$m = \frac{T}{kJ} \frac{\mathrm{d}}{\mathrm{d}m} \int r\,\mathrm{d}r \frac{\mathrm{d}\theta}{2\pi} \exp(-r^2/2) \ln \mathrm{I}_0(\Xi)$$

$$\Xi = \frac{J}{T}(\kappa^2 m^2 + \tfrac{1}{2} q r^2 + \sqrt{2}\kappa m q^{1/2} r \cos\theta)^{1/2}. \qquad (23)$$

Analogously to the spin glass, m and q can be identified as

$$m = [\langle \cos\phi_i \rangle],$$
$$q = [(\langle \cos\phi_i \rangle)^2] \qquad (24)$$

where brackets $\langle\,\rangle$ denote the thermal average.

The solution of (23) leads to the phase diagram presented in figure 1. In the case $\kappa > 1$ (i.e. small dispersion, see (5)) the system has a transition from the disordered phase (both order parameters equal to zero) into the superconducting phase (long-range superconducting order exists: $m \neq 0$) when the temperature is lowered. When $\kappa < 1$ (large dispersion) the transition is into the superconductive glass phase ($m = 0$; $q \neq 0$). For $T = 0$, the transition from superconducting to superconductive glass occurs at $\sqrt{4/\pi}$.

In the case $\kappa = 0$, i.e. infinite dispersion, we obtain the same result as in [8], where the case was studied when A_{ij} is uniformly distributed between 0 and 2π.

Finally, we might contrast the above with related spin-glass problems. First, we note that although figure 1 is qualitatively reminiscent of the phase diagram of the Ising SK spin glass [5], the transition at $T=0$ occurs at a different value of the relevant parameter ($\kappa=\sqrt{4/\pi}$ in place of $J_0/J=\sqrt{2/\pi}$) even if the scaling is chosen to make the disordered to ordered transitions coincident in the two problems. Second, we note that the model is quite different from the planar SK spin-glass in several ways; (i) at the level of the Hamiltonian, where (1) lacks the invariance under the global gauge transformation $\phi_i \rightarrow -\phi_i$ all i, which is a feature of the spin-glass, (ii) in the form of the effective replicated Hamiltonian, where the spin glass has an interaction term $T_i^{\alpha\beta} \cdot T_j^{\alpha\beta}$ with $T_i^{\alpha\beta}$ a unit vector of turn angle $\psi_i^{\alpha\beta}=\phi_i^{\alpha}+\phi_i^{\beta}$, as well as the $S_i^{\alpha\beta} \cdot S_j^{\alpha\beta}$ found in both models, and (iii) in the phase diagram structure where the spin-glass has four regions corresponding to paramagnet, isotropic spin-glass, collinear ferromagnet, and mixed ferromagnet and transverse spin-glass (canted ferromagnetic glass).

We thus see that, although relatively straightforward to analyse by spin-glass inspired techniques, the model discussed here represents a new situation by comparison with spin-glasses. The extent to which it mimics features of a real granular superconductor or a short-range gauge glass remains less clear.

The work of M Simkin has been supported by the Soros Foundation and by the Russian Foundation for Fundamental Research (Grant 93-02-2535).

References

[1] Ebner C and Stroud D 1985 *Phys. Rev.* B **31** 165
[2] Morgenstern I, Muller K A and Bednorz J G 1987 *Z. Phys.* B **69** 33
[3] Aksenov V L and Sergeenkov S A 1988 *Physica* **156C** 18
[4] Fisher M P A 1989 *Phys. Ref. Lett.* **62** 1415
[5] Sherrington D and Kirkpatrick S 1979 *Phys. Rev. Lett.* **35** 1972
[6] Sherrington D 1992 *Electronic Phase Transitions* ed W Hanke and Yu V Kopaev (Amsterdam: Elsevier) p 79
[7] Kirkpatrick S and Sherrington D 1978 *Phys. Rev.* B **17** 4384
[8] Moore M A and Murphy S unpublished